

Many-body localization identified by entanglement in momentum space

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(Dated: August 16, 2016)

We study the entanglement in momentum space of a disordered one-dimensional fermion lattice model with attractive interaction. We observe that the many-body localization transition can be characterized by behaviors of two components in entanglement spectrum. One of the components is related to paired-fermion entanglement which contributes to the long-range correlation in position space, and the vanishing of it indicates the emerged many-body localized phase. Based on this understanding, we identify the critical point between delocalized and localized phases for interaction of different strength. Additionally by method of entanglement spectrum, we provide a new evidence to show the transition of two phases induced by interaction, and find this phase transition is not influenced by the disorder. Our result shows key characteristics in entanglement for different phases in the system, and provides a novel perspective to understand many-body localization phenomenon.

PACS numbers: 03.67.Mn, 64.70.Tg, 71.23.An, 71.10.Fd

Introduction— It is known that a vanishing density of states at fermi level leads to an insulator in band theory of condensed matter physics. However, the well-known Anderson localization induced by disorder for noninteracting particles shows a different mechanism of insulation [1, 2]. Recently, there is a revival interest in localization due to disorder in the presence of interaction, known as many-body localization (MBL) [3–5]. The MBL may protect a closed quantum many-body system from thermalization which questions our fundamental assumption in statistical physics, suggesting some emergent conservation laws for those localized systems [6]. The localized phase can be closely related with concepts like eigenstate thermalization hypothesis [7, 8], the area-law of entanglement entropy [9–12]. The systems showing MBL phenomenon may have a transition between localized and delocalized phases depending on disorder strength in the Hamiltonian [13, 14], and many breakthroughs have been made [15–17].

Quantum entanglement, a unique feature in quantum physics, is shown to be a powerful tool in illustrating the quantum characteristics of many-body systems. For a pure bipartite state, entanglement spectrum (ES) shows more information of entanglement than entanglement entropy which is a measure of entanglement [18]. The ES can be obtained by finding the logarithm of eigenvalues of reduced density matrix of a subsystem for a many-body quantum state partitioned into two parts. By investigating the ES in position space, the method has been successfully applied to investigate fractional quantum Hall states [18, 19], complex paired superfluids [20], and spin-orbit coupled superconductors [21], etc. Also, ES in momentum space [22–26] and angular momentum space [27] can help to explore new phases of quantum matters of many-body systems such as Chern insulators.

Closely related with ES, the entanglement can also be quantified by the Rényi entropy of the reduced density matrix, which can characterize the local convertibility of the bipartite quantum states and is applicable in quantum phase transition [28].

In this Letter, we focus on a fermion lattice model with nearest-neighbor interaction and on-site disorder. This system is known to have MBL phase and some other quantum phases [29–32]. The entanglement in position space has been investigated in spin-1/2 Heisenberg model, which is equivalent to a special case of this fermion model, and shows a phase transition from delocalized to localized phases [33, 34]. Different phases come up depending on strength of interaction and disorder. However, there is uncertainty in the phase transition critical point, and further evidences are necessary. Here, our work will concentrate on the entanglement in momentum space to analyze this fermion model. When investigating the entanglement between particles with positive and negative momentum, we depict the features and intuitive picture of two components of it: one component related to the entanglement between particles with opposite momentum would be destroyed by the other component as the strength of disorder grows, which indicates MBL phenomenon of this model. When investigating the entanglement between particles with small and large momentum, we confirm the phase transition that origins from the Hamiltonian without disorder, and we show that the behavior of this transition is robust with the presence of on-site disorder.

Model and method— We consider a spinless one-dimensional fermion model with attractive nearest-neighbor interaction and on-site random disorder. The

Hamiltonian of the system is written as,

$$H = -t \sum_{i=1}^L (c_i^\dagger c_{i+1} + H.c.) + \sum_{i=1}^L h_i c_i^\dagger c_i + U \sum_{i=1}^L (c_i^\dagger c_i - \frac{1}{2})(c_{i+1}^\dagger c_{i+1} - \frac{1}{2}), \quad (1)$$

in which L is the number of sites in the chain, the c_i^\dagger and c_i are the creation and annihilation operators on site i , $U < 0$ represents attractive interaction, and h_i randomly but uniformly distributed in $[-W/2, W/2]$ represents on-site disorder. We let $t = 1$ for simplicity. We remark that when $U = -1$, the model is equivalent to spin-1/2 Heisenberg model by applying Jordan-Wigner transformation [35].

In order to investigate the ES in momentum space, we transform the Hamiltonian into particle number representation in momentum space, which means that periodic boundary condition is assumed. The Hamiltonian in momentum space, up to a constant, is written as,

$$H = -\frac{2}{\sqrt{L}} \sum_p \cos(\frac{2p}{L}\pi) a_p^\dagger a_p + \sum_{p,q} g_{p-q} a_p^\dagger a_q + \frac{4U}{L} \sum_{\substack{p+q=p'+q' \\ p>q, p'>q'}} \sin(\frac{p-q}{L}\pi) \sin(\frac{p'-q'}{L}\pi) a_{p'}^\dagger a_p a_{q'}^\dagger a_q, \quad (2)$$

in which g_p is Fourier transform of h_i , a_p^\dagger is creation operator of fermion with pseudo-momentum $2\pi p/L$, and the summation is performed in the first Brillouin zone.

Considering the long-range interaction in the reciprocal lattice, the DMRG method may not be suitable for solving the ground state of this model in particle number representation of momentum space [36], so we use Arnoldi method to acquire it [37, 38]. Then, we use two ways to divide system into two parts in the reciprocal lattice: to investigate the entanglement between particles with positive and negative momentum requires to divide the lattice into left and right parts, and the inner-outer division demonstrates the small-large momentum entanglement.

Since the total particle number operator commutes with the Hamiltonian, we can focus on the ground state in the subspace of certain particle number N . Hence, when we divide the system, the reduced density matrix of its subsystem must be in block-diagonal form, and every block corresponds to a number n representing the amount of remaining particles in the subsystem. Thus, we can use it as a parameter for each spectral line in ES.

Entanglement of positive-negative momentum— We first investigate the ES by dividing the system into positive and negative momentum parts. We primarily concentrate on the ES in the sector corresponding

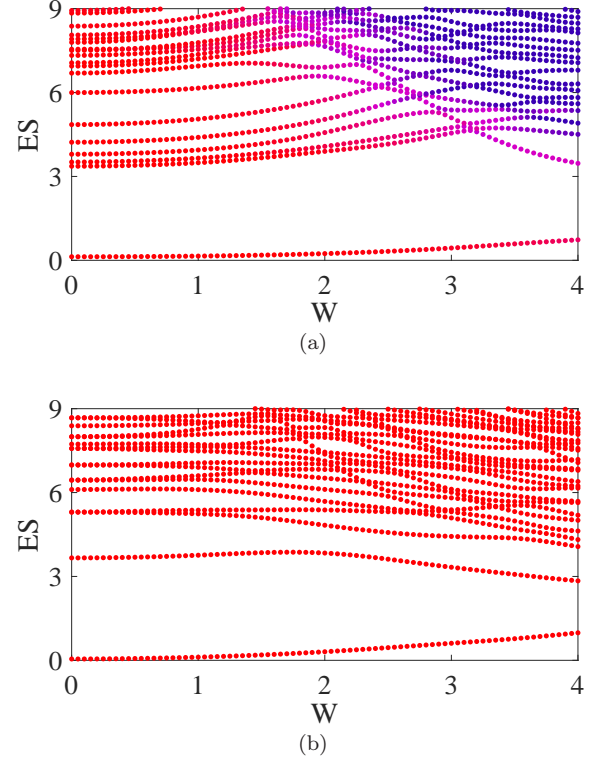


FIG. 1: ES between the positive and negative momentum parts of the system, of which the remaining particle number $n = 5$. The interaction $U = 1$ and the disorder W is the horizontal axis, with the chain length $L = 22$ and the total particle number $N = 11$. (a) reserve all scattering terms originating from nearest-neighbor interaction, the color is mapped from relative entropy of coherence, which helps to distinguish paired-particle and chaotic component; (b) only reserve scattering terms between particles with exactly opposite momentum, the spectral lines are colored uniformly red.

to that the remaining particles in positive momentum parts equals to a half of total particles (except the zero-momentum particle). For a fixed U , the ES according to different disorder strength W are shown in Fig. 1(a). Apart from the lowest one spectral line corresponding to the ground state without interaction or disorder, two components can be seen in ES. Based on the fact that the states corresponding to the spectral lines have different coherence for different component, it helps us distinguish the two components to map the colors of each line from relative entropy of coherence, which can quantify coherence of the states[39]. One component, named paired-particle component, is the main ES when W is small. As W increases, the other component, named chaotic component, goes lower, showing that it becomes more important. The descending chaotic component crosses with the original second lowest spectral line at the turning point of the second lowest spectral line, showing that the chaotic component completely destroy the paired-particle component.

We also study the Rényi entanglement entropy (REE) in the same sector. It is defined as $S_\alpha = \frac{1}{1-\alpha} \log \sum_i \lambda_i^\alpha$, in which α is a positive parameter and λ_i of different W can be got from Fig. 1(a). As shown in Fig. 2(a), the REE reaches a minimum for all α , which indicates the relative importance of different entanglement components changes. To illustrate this picture clearer, we investigate the sign of $\partial_W S_\alpha$, which shows how the strength of entanglement varies as W changes, as shown in Fig. 2(c). When fixing W to be smaller than a certain value, the change of the signs of $\partial_W S_\alpha$ for different α 's shows there is no global shift of all S_α . Since different α 's in Rényi entropy represents different scales to measure entanglement, two directions of shift in S_α with different α 's mean there are two components of entanglement and they changes in different ways as W increases. Therefore, considering that the chaotic component becomes more important as W increases, only when all $\partial_W S_\alpha$ have the same sign at a certain W , paired-particle component is

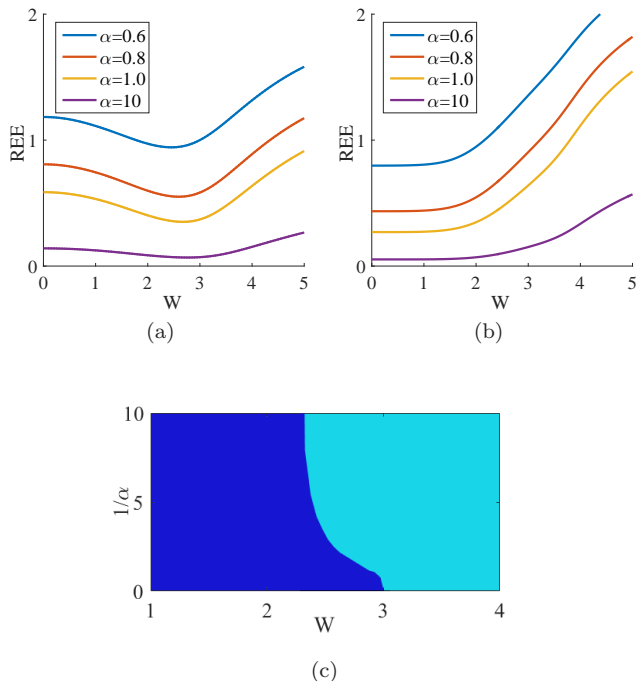


FIG. 2: The behavior of REE between the positive and negative momentum parts of the system, of which the remaining particle number $n = 5$. The interaction $U = 1$ and the disorder W is the horizontal axis, with the chain length $L = 22$ and the total particle number $N = 11$. Rényi entropy for different α when (a) reserving all scattering terms originating from nearest-neighbor interaction, (b) only reserving scattering terms between particles with exactly opposite momentum. (c) The sign distribution of $\partial_W S_\alpha$ on the $1/\alpha - W$ plane. $\partial_W S_\alpha$ is positive in cyan (light gray) regions and negative in the blue (dark gray) regions. When the sign takes on both positive and negative at certain W , paired-particle component of entanglement still exists. Otherwise, it is completely destroyed.

completely destroyed. This value of W , which will be identified later, indicates the critical point of quantum phase transition [28].

Then, we will show that the so-called paired-particle component indeed originates from entanglement between particle pairs with opposite momentum, and is related with long-range correlation.

It is known that the attractive interaction may cause fermions to pair, and the interaction terms relating fermions with opposite momentum plays a central role in this system. Similar to BCS theory of superconductivity, if we only reserve those terms relating particles with exact opposite momentum and drop other scattering terms, then perform Bogoliubov transformation, we can acquire a set of newly defined pseudo-particles which can be seen as the linear combination of the original fermions with opposite momentum [40, 41]. Hence the eigenstate of the system can be solved in the single-particle picture of these pseudo particles. The ground state, as well as excited state, is exactly the basis of particle number representation of pseudo-particles. These pairs represent the entanglement between particles with opposite momentum, which induces long-range correlation. Also, the pairs, which could not be destroyed when adding disorder (only with exactly opposite momentum scattering terms), would contribute to p-wave superconductivity [32, 42].

According to the above consideration, the reserved interaction term could be written as,

$$V = \frac{4U}{L} \sum_{p,p' > 0} \sin\left(\frac{2p}{L}\pi\right) \sin\left(\frac{2p'}{L}\pi\right) a_{p'}^\dagger a_p^\dagger a_{-p'} a_{-p}, \quad (3)$$

Since ES can reflect entanglement in detail, the ES of the ground state of the Hamiltonian with this interaction can be used as a signature of the long-range correlation. We investigate this ES as shown in Fig. 1(b). The absence of turning point in the second lowest spectral line in Fig. 1(b) suggests that no chaotic component destroy the original component in ES relating to pair-particle entanglement. Furthermore, the absence of the chaotic component in this case demonstrates that it is generated by the scattering term between particles with inexactly opposite momentum. Also, REE reflects the same picture. As shown in Fig. 2(b), S_α monotonically increases for all α , which indicates that there is only paired-particle component of entanglement when only reserving interaction term between opposite momentum.

Thus, the physical picture is clear. When disorder is weak, the paired-particle component plays the main role, suggesting that the system is delocalized. As the disorder becomes stronger, although the disorder does not destroy paired-particle component directly, it makes the chaotic component more important and decisive, which means the long-range correlation is destroyed and the system becomes localized. We have already seen in Fig. 1(a)

that as the chaotic component goes lower, the position of the second lowest line would meet a turning point when the paired-particle component loses its leading role. This point, along with the vanishing of red region in 2(c) which occurs slightly earlier than this point due to the effect of the lowest spectral line, indicates the transition from delocalized to localized phase. Since this picture is based on the modes of pseudo-particle which are particle pairs, and does not depend on whether these modes are excited, we claim that the transition occurs to not only ground state but also not highly excited states of the system. At $U = -1$, we studied 20 samples and find this point occurs at $W_c = 3.2 \pm 0.5$, which is consistent with previous studies [33, 43, 44]. When $U \in [-1, -2)$, we find the same picture still applies, so we investigated 10 samples and acquire the phase diagram of this region as shown in Fig. 3.

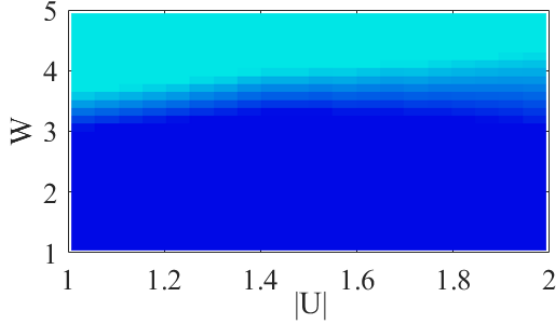


FIG. 3: The phase diagram when the interaction $U \in [-1, -2)$ with the chain length $L = 20$ and the total particle number $N = 10$. The cyan (light gray) region is the localized phase; the blue (dark gray) region is the delocalized phase. The uncertainty due to finite size with random disorder is shown by gradient colors.

Entanglement of small-large momentum— In this case, we first investigate ES of the system without disorder. We divide system into small and large momentum parts along the fermi level of non-interacting case. When adding interaction, the particles would be scattered to above the fermi level, leaving holes below. Thus, the entanglement between these two parts can be seen as entanglement between particles and holes. As is shown in Fig. 4, in each sector of which remaining particle number n equals to total particle number N minus an even number, there is a gap between the lowest spectral line and upper parts. As $|U|$ increases, the gap increases to reach a sharp maximum when $U = -2$, and then decreases. Since the apparent entanglement between different sectors is due to that the particle number is a good quantum number and thus does not fundamentally reflects the entanglement between particles and holes, the only one spectral line in every individual sector shows the vanishing of their entanglement, which suggests a transition from one structure of entanglement to another and

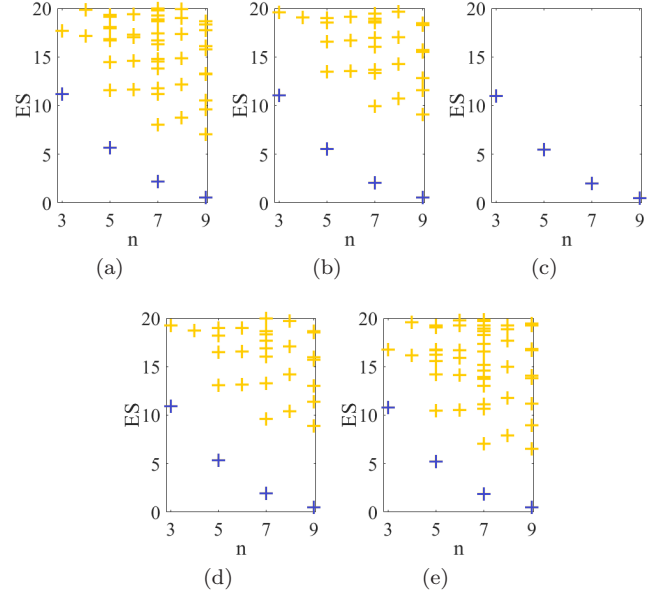


FIG. 4: ES between small and large momentum parts of the system, when there is no disorder. The chain length $L = 22$ and total particle number $N = 11$. The blue (dark gray) lines represent the lowest spectrum, and the orange (light gray) lines represent the upper parts. The strength of interaction U equals to (a)1.97, (b)1.99, (c)2.00, (d)2.01, (e)2.03.

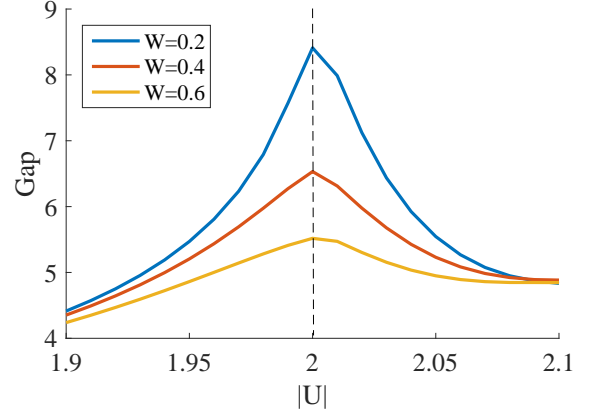


FIG. 5: The gap in the ES, of which the remaining particle number $n = 9$. The chain length $L = 22$ and the total particle number $N = 11$. Different colors (gray levels) represent different disorders W .

thus resulting in a probable phase transition. This is consistent with the conclusion achieved by typical other methods [45, 46].

When there is disorder but not too large, the gap in the ES according to different U is shown in Fig. 5. Although the entanglement between particles and holes does not completely vanish, it gets to be the weakest and thus the gap reaches the maximum still at $U = -2$, which shows the robustness of the critical point when adding disorder [30, 32].

Conclusion— By partitioning the reciprocal lattice into two subsystems in different ways, we investigate the entanglement in momentum space of a fermion lattice model with interaction and disorder. When partitioning particles with positive and negative momentum, we find that the ES consists of two components. Comparing with the ES of system possessing interaction only between particles with opposite momentum, we show that one of the component is originated from paired-particle entanglement, i.e. the entanglement between particles with opposite momentum, and this entanglement represents long-range correlation in position space. However, the other component, chaotic entanglement, may destroys the paired-particle entanglement and induces MBL phenomenon. Based on this picture, we obtain the critical point between the delocalized phase and localized phase of this model. When dividing the momentum space into small and large momentum parts, we observe that the behavior of the gap in ES can be used to identify a phase transition in this system, and can show that this transition is irrelevant to disorder.

Our work provides a clear picture in understanding disordered fermion lattice model from a novel perspective - by investigating entanglement in momentum space. The nature and characteristics of many-body localization and the related phase transition can be studied by methods of entanglement such as standard entanglement entropy, entanglement spectrum, Rényi entanglement entropy and its derivative. Those methods can also be applicable to many-body localization in other systems.

This work is supported by MOST of China (2016YFA0302104, 2016YFA0300600), NSFC (91536108), NFFTBS (J1030310, J1103205), the Strategic Priority Research Program of the Chinese Academy of Sciences (XDB01010000, XDB21030300), the Young Elite Program for Faculty of Universities in Beijing, and Training Program of Innovation for Undergraduates of Beijing.

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